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# Stability of Equilibria for a Satellite Subject to Gravitational and Constant Torques

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**We study the stability of equilibria of a rigid body that moves along a circular orbit and is subject to gravity-gradient and constant torques. For each orientation of the satellite, there exists a constant torque that provides an equilibrium. For two important special cases, stability can be studied analytically. When one of the satellite's central principal axes is aligned with one of the axes of the orbital reference frame, the necessary conditions of stability are satisfied for appropriate values of inertial parameters. When one of the principal axes lies in a coordinate plane of the orbital frame, equilibria are proved to be unstable. In the general case, the stability is studied numerically.**

## Introduction

ANALYSIS of a satellite's steady-state motion is essential for the development of active and passive attitude control systems. Equilibria of a satellite with respect to the orbital reference frame are frequently used as nominal motions in the design of attitude control systems. In an inertial frame, such a relative equilibrium corresponds to a precession of the satellite, that is, a rotation of the satellite about the normal to the orbit plane with constant orbital angular velocity. This motion exists when the total torque applied to the satellite causes the appropriate change of its angular momentum.

Such equilibria were studied for various combinations of torques and dynamical properties of a satellite. For a rigid body in the central gravitational field this was done in [1–3]. Equilibrium orientations of a satellite gyrostat were analyzed in [4–7]. Equilibria of a satellite acted upon by gravitational and aerodynamic torques were examined in [8].

In this paper we analyze the behavior of a satellite in a circular orbit when it is subject to the influence of gravitational and constant torques, the latter either produced actively or resulting from gas or fuel leakages. It is known that the influence of a small constant torque on satellite dynamics is similar to the action of the nonconservative component of the aerodynamic torque [9–11].

There are several publications that focus on the dynamics of a rigid body subject to a constant torque (see [12–16]). However, these papers do not take into account the action of the gravity-gradient torque, so the results of [12–16] provide only an approximate description of the satellite's orbital dynamics (unless the satellite is spherically symmetric).

Next we consider a satellite subject to both constant and gravity-gradient torques. The first results concerning this problem were obtained in 1963 by Garber who found some stable equilibrium configurations [17]. The next step in the analysis of the action of both gravity-gradient and constant torques was done in [18–20], where we described the number of existing equilibria as a function of all components of a constant torque. For a small constant torque, there exist 24 equilibria; when the torque is sufficiently large, there are none. In the general case, the number of equilibria is a fairly complicated function of constant torque components, and it was shown that a decrease of the torque magnitude may lead to an increase of the number of equilibrium orientations.

In this paper, we study the stability of equilibrium orientations. We proceed from the following assumptions: 1) the gravity field of the Earth is central and Newtonian; 2) the satellite is a triaxial rigid body; 3) the satellite's center of mass moves along a circular orbit; 4) the satellite is subject to the gravity-gradient torque and a torque that is fixed with respect to the body of the satellite, so the components of this torque in the body-fixed frame are constant. The particular feature of this problem is that the equations of the satellite's motion under the above assumptions (unlike, for example, those for the gyrostat satellite) do not admit any integral of motion, so we can verify only the necessary stability conditions.

Here we study analytically two special cases. In the first case, one of the satellite's central principal axes of inertia is directed along an axis of the orbital reference frame. In the second special case, one of the satellite's central principal axes of inertia lies in a coordinate plane of the orbital frame.

We also conduct a numerical analysis of the general case of the satellite's equilibrium orientation.

## Statement of the Problem

Two right-hand reference frames related to the center of mass of the satellite $O$ are used to study stability of relative equilibria of a satellite acted upon by gravitational and constant torques.

In the orbital coordinate system $OX_1X_2X_3$, the axis $OX_1$ is directed along the velocity of the satellite, axis $OX_3$ lies along the radius vector of the point $O$ with respect to the center of the Earth, and $OX_2$ is normal to the orbit.

In the frame $Ox_1x_2x_3$ connected with the satellite $Ox_1$, $Ox_2$ and $Ox_3$ are its central principal axes. The corresponding moments of inertia are $A$, $B$, and $C$.

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The mutual orientation of these frames can be described by the orthogonal matrix $A = [a_{ij}]$ where the elements $a_{ij}$ are the direction cosines of the axes of the $Ox_1x_2x_3$ system with respect to $OX_1X_2X_3$. As usual,

$$a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3} = \delta_{ij}, \qquad i, j = \overline{1, 3} \tag{1}$$

where $\delta_{ij}$ is the Kronecker symbol. The transition from system $OX_1X_2X_3$ to system $Ox_1x_2x_3$ can be realized by three Euler's rotations about axes 2, 3, and 1 through the angles $\alpha$ (pitch), $\beta$ (yaw), and $\gamma$ (roll). Matrix $A$ is expressed through these angles as

$$A = \begin{pmatrix} \cos\alpha\cos\beta & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma & \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma \\ \sin\beta & \cos\beta\cos\gamma & -\cos\beta\sin\gamma \\ -\sin\alpha\cos\beta & \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma \end{pmatrix} \tag{2}$$

Obviously, the two sets of orientation angles $(\alpha, \beta, \gamma)$ and $(\alpha + \pi, \pi - \beta, \gamma + \pi)$ correspond to the same orientation matrix. For this reason, we consider only the values $0 \le \alpha < 2\pi$, $-\pi/2 \le \beta \le \pi/2$, and $0 \le \gamma < 2\pi$. Note that the indicated system of orientation angles degenerates for $\beta = \pi/2$ and $\beta = -\pi/2$. In this case the orientation of the satellite can be described by the combination of the other two angles, $\alpha + \gamma$ and $\alpha - \gamma$, respectively.

Under the previous assumptions, the equations of the satellite's attitude motion can be written in the following form [18]:

$$\begin{aligned} A\dot{\tilde{p}} + (C - B)\tilde{q}\,\tilde{r} &= 3\omega_0^2(C - B)a_{32}a_{33} + \tilde{a} \\ B\dot{\tilde{q}} + (A - C)\tilde{r}\,\tilde{p} &= 3\omega_0^2(A - C)a_{33}a_{31} + \tilde{b} \\ C\dot{\tilde{r}} + (B - A)\tilde{p}\,\tilde{q} &= 3\omega_0^2(B - A)a_{31}a_{32} + \tilde{c} \\ \tilde{p} = \dot{\alpha}a_{21} + \dot{\gamma} + \omega_0 a_{21}, &\qquad \tilde{q} = \dot{\alpha}a_{22} + \dot{\beta}\sin\gamma + \omega_0 a_{22} \\ \tilde{r} &= \dot{\alpha}a_{23} + \dot{\beta}\cos\gamma + \omega_0 a_{23} \end{aligned} \tag{3}$$

Here $\tilde{p}$, $\tilde{q}$, and $\tilde{r}$ are the projections of the satellite's angular velocity on the axes of the $Ox_1x_2x_3$ frame, while $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ are the components of the constant torque in the same frame.

The equilibrium orientations of the satellite $\alpha = \alpha_0$, $\beta = \beta_0$, and $\gamma = \gamma_0$, where $\alpha_0$, $\beta_0$, and $\gamma_0$ are constants, correspond to solutions of the system

$$\begin{aligned} a_{22}a_{23} - 3a_{32}a_{33} = a, &\qquad a_{23}a_{21} - 3a_{33}a_{31} = b \\ a_{21}a_{22} - 3a_{31}a_{32} = c, &\qquad a_{21}^2 + a_{22}^2 + a_{23}^2 = 1 \\ a_{31}^2 + a_{32}^2 + a_{33}^2 = 1, &\qquad a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0 \end{aligned} \tag{4}$$

The dimensionless components of the constant torque used in Eq. (4) are

$$a = \frac{\tilde{a}}{\omega_0^2(C - B)}, \qquad b = \frac{\tilde{b}}{\omega_0^2(A - C)}, \qquad c = \frac{\tilde{c}}{\omega_0^2(B - A)}$$

We assume that for the satellite $A \ne B$, $B \ne C$, and $A \ne C$. [Note that if the satellite is axisymmetric, i.e., $A = B$, the equilibria exist only for $\tilde{c} = 0$. Then for each set $(\tilde{a}, \tilde{b}, 0)$, there appears a one-parametric family of solutions of Eq. (4).]

System (4) consists of six equations for six unknown quantities $a_{21}$, $a_{22}$, $a_{23}$, $a_{31}$, $a_{32}$, and $a_{33}$. When these are found, the direction cosines $a_{11}$, $a_{12}$, and $a_{13}$ can be determined from conditions (1).

The system that describes equilibrium orientations of the satellite in terms of the orientation angles $\alpha_0$, $\beta_0$, and $\gamma_0$ is obtained by substituting in Eqs. (4) the expressions of direction cosines $a_{ij}$ from Eq. (2):

$$\begin{aligned} &3(\cos\alpha_0\sin\gamma_0 + \sin\alpha_0\sin\beta_0\cos\gamma_0)(\sin\alpha_0\sin\beta_0\sin\gamma_0 \\ &\quad - \cos\alpha_0\cos\gamma_0) - \cos^2\beta_0\sin\gamma_0\cos\gamma_0 = a \\ &3\sin\alpha_0\cos\alpha_0\cos\beta_0\cos\gamma_0 \\ &\quad - \sin\beta_0\cos\beta_0(1 + 3\sin^2\alpha_0)\sin\gamma_0 = b \\ &\sin\beta_0\cos\beta_0(1 + 3\sin^2\alpha_0)\cos\gamma_0 \\ &\quad + 3\sin\alpha_0\cos\alpha_0\cos\beta_0\sin\gamma_0 = c \end{aligned} \tag{5}$$

System (5) shows that for any values of $\alpha_0$, $\beta_0$, and $\gamma_0$, that is, for each orientation of the satellite, there is a unique choice of quantities $a$, $b$, and $c$ that makes this orientation an equilibrium. We study the stability of these equilibria.

## Conditions of Stability

After linearization of system (3) in the vicinity of equilibrium $\alpha = \alpha_0$, $\beta = \beta_0$, and $\gamma = \gamma_0$, one obtains the respective characteristic equation

$$a_0\lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 = 0 \tag{6}$$

where

$$\begin{aligned} a_0 &= ABC\cos\beta_0, \qquad a_1 = 0 \\ a_2 &= F_2(\alpha_0, \beta_0, \gamma_0, A, B, C)\cos\beta_0 \\ a_3 &= F_3(\alpha_0, \beta_0, \gamma_0, A, B, C)\cos\beta_0 \\ a_4 &= F_4(\alpha_0, \beta_0, \gamma_0, A, B, C)\cos\beta_0 \\ a_5 &= F_5(\alpha_0, \beta_0, \gamma_0, A, B, C)\cos\beta_0 \\ a_6 &= \tfrac{3}{4}(B - A)(A - C)(B - C)[8\cos 2\alpha_0(\cos\beta_0 \\ &\quad + \cos 3\beta_0)\cos 2\gamma_0 + \sin 2\alpha_0(2\sin 2\beta_0 - 3\sin 4\beta_0)\sin 2\gamma_0] \end{aligned}$$

The formulas defining $F_2$, $F_3$, $F_4$, and $F_5$ are rather cumbersome, so we omit them. The general form of Eq. (6) is represented in Appendix A.

Since $a_1 = 0$, the necessary condition of stability is that all the roots of Eq. (6) must be purely imaginary. Thus we arrive at the relations

$$a_3 = 0 \tag{7}$$

$$a_5 = 0 \tag{8}$$

If conditions (7) and (8) are satisfied, the characteristic equation assumes the form

$$a_0\lambda^6 + a_2\lambda^4 + a_4\lambda^2 + a_6 = 0 \tag{9}$$

Its roots are purely imaginary if and only if all the roots of the equation

$$a_0\mu^3 + a_2\mu^2 + a_4\mu + a_6 = 0 \tag{10}$$

are real and negative. By the condition of Cardano, the solutions of Eq. (10) are real when

$$Q = \left(\frac{\xi}{3}\right)^3 + \left(\frac{\eta}{2}\right)^2 \leq 0 \tag{11}$$

where

$$\xi = -\frac{1}{3}\left(\frac{a_2}{a_0}\right)^2 + \frac{a_4}{a_0}, \qquad \eta = \frac{2}{27}\left(\frac{a_2}{a_0}\right)^3 - \frac{a_2 a_4}{3a_0^2} + \frac{a_6}{a_0}$$

In this case they are negative under the Routh and Hurwitz conditions for Eq. (10), that is, when

$$a_2 a_0 > 0 \tag{12}$$

$$a_2 a_4 - a_0 a_6 > 0 \tag{13}$$

$$a_6 a_0 > 0 \tag{14}$$

System of Eqs. (7), (8), and (11–14) represents the necessary conditions of stability for the equilibria considered.

## Method of Study

The problem in study contains nine parameters: the moments of inertia of the satellite $A$, $B$, and $C$, the equilibrium orientation angles $\alpha_0$, $\beta_0$, and $\gamma_0$, and the parameters $a$, $b$, and $c$ that characterize the constant torque. The parameters $a$, $b$, and $c$ are determined by the choice of equilibrium. Given the orientation angles, they can be calculated either from systems (4) or (5). Instead of the moments of inertia, we use two sets of inertial parameters, either

$$\theta_A = A/B \quad \text{and} \quad \theta_C = C/B \tag{15}$$

or

$$x = \frac{B-A}{C} \quad \text{and} \quad y = \frac{C-B}{A} \tag{16}$$

so that $x = (1-\theta_A)/\theta_C$, $y = (\theta_C - 1)/\theta_A$, and $\theta_A = (1-x)/(1+xy)$, $\theta_C = (1+y)/(1+xy)$. For all triaxial bodies these parameters have to satisfy some obvious conditions. For parameters (15) these conditions are

$$\theta_A > 0, \qquad \theta_C > 0, \qquad \theta_A \neq \theta_C, \qquad \theta_A \neq 1, \qquad \theta_C \neq 1$$

and

$$\theta_A + \theta_C - 1 \geq 0, \qquad \theta_A - \theta_C + 1 \geq 0, \qquad -\theta_A + \theta_C + 1 \geq 0 \tag{17}$$

For a thin plate, one of the relations in Eq. (17) becomes an equality. The restrictions for parameters (16) can be written as

$$\begin{gathered} -1 \leq x \leq 1, \qquad -1 \leq y \leq 1, \qquad x \neq 0 \\ y \neq 0, \qquad x + y \neq 0 \end{gathered} \tag{18}$$

One can also notice that the value $x = 1$ implies $y = -1$, and vice versa. This case corresponds to a thin plate with $B = A + C$. For $x = 1$ and/or $y = -1$, this set of inertia parameters degenerates. The values $x = -1$ and $y = 1$ also correspond to a thin plate (with $A = B + C$ and $C = A + B$, respectively). For this reason, in our analysis of three-dimensional satellites we use parameters (16) satisfying restrictions (18) and assume that

$$-1 < x < 1, \qquad -1 < y < 1 \tag{19}$$

To study thin plates, we use parameters (15).

Thus, the problem can be described in terms of five parameters, namely, two inertial parameters, either (15) or (16), and three orientation angles $\alpha_0$, $\beta_0$, $\gamma_0$ of Eq. (2).

To study stability of equilibria, one can use the following procedure: 1) to eliminate two parameters using Eqs. (7) and (8); 2) to verify conditions (11–14) in space of the remaining three parameters.

Three different situations were detected during the analysis of Eqs. (7) and (8).

In special case I, one can indicate some values of parameters for which the coefficients $a_3$ and $a_5$ vanish.

In special case II, we manage to resolve the system of Eqs. (7) and (8) analytically, which simplifies the analysis of conditions (11–14).

In the general case, we had to make use of a numerical procedure to eliminate two parameters from Eqs. (7) and (8) and then verify Eqs. (11–14).

## Special Case I

The analysis of Eqs. (7) and (8) shows that this system is satisfied for three types of equilibria.

The first group of equilibria that satisfy Eqs. (7) and (8) corresponds to

$$a \neq 0, \qquad b = 0, \qquad c = 0 \tag{20}$$

and consists of the following solutions of system (5): either

$$\sin\alpha_0 = 0, \qquad \sin\beta_0 = 0 \tag{21}$$

or

$$\cos\alpha_0 = 0, \qquad \sin\beta_0 = 0 \tag{22}$$

or

$$\cos\beta_0 = 0 \tag{23}$$

In the last case $\beta_0 = \pm\pi/2$, so the chosen system of orientation angles degenerates, as well as the characteristic equation.

The second group of solutions of system (5) that satisfy Eqs. (7) and (8) corresponds to

$$a = 0, \qquad b \neq 0, \qquad c = 0 \tag{24}$$

and consists of the solutions

$$\cos\alpha_0 = 0, \qquad \cos\gamma_0 = 0 \tag{25}$$

or

$$\sin\alpha_0 = 0, \qquad \cos\gamma_0 = 0 \tag{26}$$

or

$$\sin\beta_0 = 0, \qquad \sin\gamma_0 = 0 \tag{27}$$

In case (27) an additional condition should be imposed on the inertial parameters of the satellite:

$$\theta_A + \theta_C = 1$$

Finally, the third group of equilibrium orientations that satisfy Eqs. (7) and (8) corresponds to

$$a = 0, \qquad b = 0, \qquad c \neq 0 \tag{28}$$

and consists of the solutions

$$\cos\alpha_0 = 0, \qquad \sin\gamma_0 = 0 \tag{29}$$

or

$$\sin\alpha_0 = 0, \qquad \sin\gamma_0 = 0 \tag{30}$$

or

$$\sin\beta_0 = 0, \qquad \cos\gamma_0 = 0 \tag{31}$$

For solution (31) the stability is also possible only if the satellite is a thin plate and the inertial parameters obey the condition

$$\theta_C - \theta_A = 1$$

We use the term initial orientation (IO) to refer to an orientation of the satellite when its principal axes are directed along the axes of the orbital coordinate system (so there are 24 possible initial orientations). In all the above special cases but case (23), one can easily see that the equilibrium orientation can be obtained by rotation of the satellite through one of the angles $\alpha_0$ (pitch), $\beta_0$ (yaw), or $\gamma_0$ (roll) starting from an IO. When this rotation is performed through the yaw angle $\beta_0$ [cases (25), (26), (29), and (30)] or the roll angle $\gamma_0$ [cases (21) and (22)], conditions (7) and (8) are satisfied for all the inertia parameters of the satellite. When the equilibrium orientation is achieved by means of the rotation through the pitch angle $\alpha_0$ [cases (27) and (31)], these conditions are satisfied only for a satellite in the form of a thin plate. In all particular cases listed above, one of the axes of the orbital reference frame coincides with a satellite's central principal axis of inertia.

As already mentioned, case (23) corresponds to degeneration of the chosen system of orientation angles. Nevertheless, we can study it in the same manner. Indeed, in this case $\beta_0 = \pm\pi/2$, and so the $Ox_1$ axis is orthogonal to the orbit plane $OX_1X_3$, and $Ox_2$ and $Ox_3$ axes lie in the $OX_1X_3$ plane. Hence, the orientation in question can be obtained from one of the IOs by a pitch rotation, and case (23) can be reduced to case (27) or (31) by an appropriate relabeling of the coordinate axes.

Because the initial orientation is irrelevant for the following analysis, we will distinguish three essentially different situations:

$$\text{(Ia)} \quad 0 \le \alpha_0 < \pi/2, \qquad \beta_0 = \gamma_0 = 0, \qquad \theta_A + \theta_C = 1$$
$$\text{(Ib)} \quad 0 \le \beta_0 < \pi/2, \qquad \alpha_0 = \gamma_0 = 0$$
$$\text{(Ic)} \quad 0 \le \gamma_0 < \pi/2, \qquad \alpha_0 = \beta_0 = 0$$

In all these special cases the characteristic equation assumes form (9). Now it is necessary to verify relations (11–14).

### Special Case Ia: $0 \le \alpha_0 < \pi/2$, $\beta_0 = \gamma_0 = 0$, $\theta_A + \theta_C = 1$

In this case the satellite is a thin plate situated in the orbit plane. The components of the constant torque are

$$a = c = 0, \qquad b = \tfrac{3}{2}\sin 2\alpha_0$$

The coefficients of the characteristic equation are

$$a_0 = (1-\theta_C)\theta_C > 0, \qquad a_2 = -\theta_C(1-\theta_C)(-5+3\xi)$$
$$a_4 = -\theta_C(1-\theta_C)(-4+15\xi), \qquad a_6 = -12\theta_C(1-\theta_C)\xi$$

where $\xi = (2\theta_C - 1)\cos 2\alpha_0$. Now we consider conditions (11–14) which in this case can be written as

$$\frac{1}{12}(4+3\xi)^2(1+3\xi)^2 \ge 0, \qquad -5+3\xi < 0$$
$$(-4+3\xi)(-1+3\xi) > 0, \qquad \xi < 0$$

respectively. This system is satisfied if and only if

$$\xi < 0$$

So conditions (11–14) are now equivalent to

$$(2\theta_C - 1)\cos 2\alpha_0 < 0$$

The necessary condition of stability is

$$\theta_C < 1/2 \quad \text{for} \quad \cos 2\alpha_0 > 0, \qquad \theta_C > 1/2 \quad \text{for} \quad \cos 2\alpha_0 < 0$$

For any triaxial thin plate there exists a set of equilibrium orientations satisfying the necessary conditions of stability. In these orientations, the plate is situated in the orbit plane, and the angle between the local vertical and the axis of the minimal moment of inertia is less then $\pi/4$.

### Special Case Ib: $0 \le \beta_0 < \pi/2$, $\alpha_0 = \gamma_0 = 0$

The components of the constant torque in this case are

$$a = 0, \qquad b = 0, \qquad c = \tfrac{1}{2}\sin 2\beta_0$$

The coefficients of the characteristic equation can be written in terms of inertial parameters $x$ and $y$ as

$$\begin{aligned}
a_0 &= \frac{(1-x)(1+y)}{(1+xy)^2}\cos\beta_0 \\
a_2 &= -\frac{(1-x)(1+y)}{2(1+xy)^3}[-2+6x-x^2+12y-2xy \\
&\quad + 6xy^2 + x^2y^2 + (x^2+2xy+x^2y^2)\cos 2\beta]\cos\beta_0 \\
a_4 &= -\frac{(1-x)(1+y)}{2(1+xy)^3}[3x-4x^2+6y-18xy-18y^2 \\
&\quad + 3xy^2 + 4x^2y^2 + (3x+4x^2+8xy-6x^2y-9xy^2 \\
&\quad + 4x^2y^2)\cos 2\beta_0]\cos\beta_0 \\
a_6 &= \frac{12(1-x)(1+y)(x+y)xy}{(1+xy)^3}\cos 2\beta_0\cos\beta_0
\end{aligned} \tag{32}$$

Figure 1 shows the regions of stability in the plane $(x, y)$ for the indicated values of $\beta_0$ (the $x$ axis is horizontal, and the $y$ axis is vertical). The white regions correspond to instability; in the dark areas the necessary conditions of stability are satisfied. The regions of stability exist for any $\beta_0 \ne \pi/4$. The value $\beta_0 = \pi/4$ is excluded because in this case the coefficient $a_6$ of the characteristic equation vanishes, and the corresponding equilibrium orientation is unstable. For $\beta_0 = 0$, Fig. 1 represents the well-known conditions of stability for a rigid body in a circular orbit [1]. For $0 < \beta_0 < \pi/4$, as well as for $\pi/4 < \beta_0 < \pi/2$, the regions of stability vary slowly (we represent here the cases $\beta_0 = \pi/12$, $\pi/6$, $\pi/4 - \varepsilon$ and $\beta_0 = \pi/4 + \varepsilon$, $\pi/3$, $5\pi/12$, $\pi/2 - \varepsilon$ where $\varepsilon$ is small). The picture changes drastically when $\beta_0$ crosses the value $\beta_0 = \pi/4$. From the mathematical point of view, the effect is pretty obvious since the essential part of coefficients (32) depends on $\cos 2\beta_0$ that changes sign for $\beta_0 = \pi/4$. Physically, the effect is related to the mass distribution of satellite and mutual orientation of the orbital frame and the frame connected with the satellite. In fact, one observes that the necessary conditions of stability for equilibria of special case Ib are always satisfied when the axis of the satellite's minimum moment of inertia is directed along the local vertical, whereas the angle between the axis of its maximum moment of inertia and the normal to the orbit plane is less then $\pi/4$ (for $0 < \beta_0 < \pi/4$ this is the region with $x > 0$, and $y < -x$, and for $\pi/4 < \beta_0 < \pi/2$ it corresponds to $x < 0$, and $y < 0$). There are also some regions of stability that correspond to equilibria with the axis of medium moment aligned with the local vertical, and the angle between the axis of maximum inertia moment and the tangent to the orbit is less then $\pi/4$ (for $0 < \beta_0 < \pi/4$ these regions belong to the domain $x < 0$, and $0 < y < -x$, whereas for $\pi/4 < \beta_0 < \pi/2$ they lie within $x > 0$, and $-x < y < 0$). In the latter case, some additional conditions have to be satisfied.

### Special Case Ic: $0 \le \gamma_0 < \pi/2$, $\alpha_0 = \beta_0 = 0$

As follows from Eq. (5), these orientations exist if

$$b = c = 0, \qquad a = -2\sin 2\gamma_0$$

The coefficients of the characteristic equation are

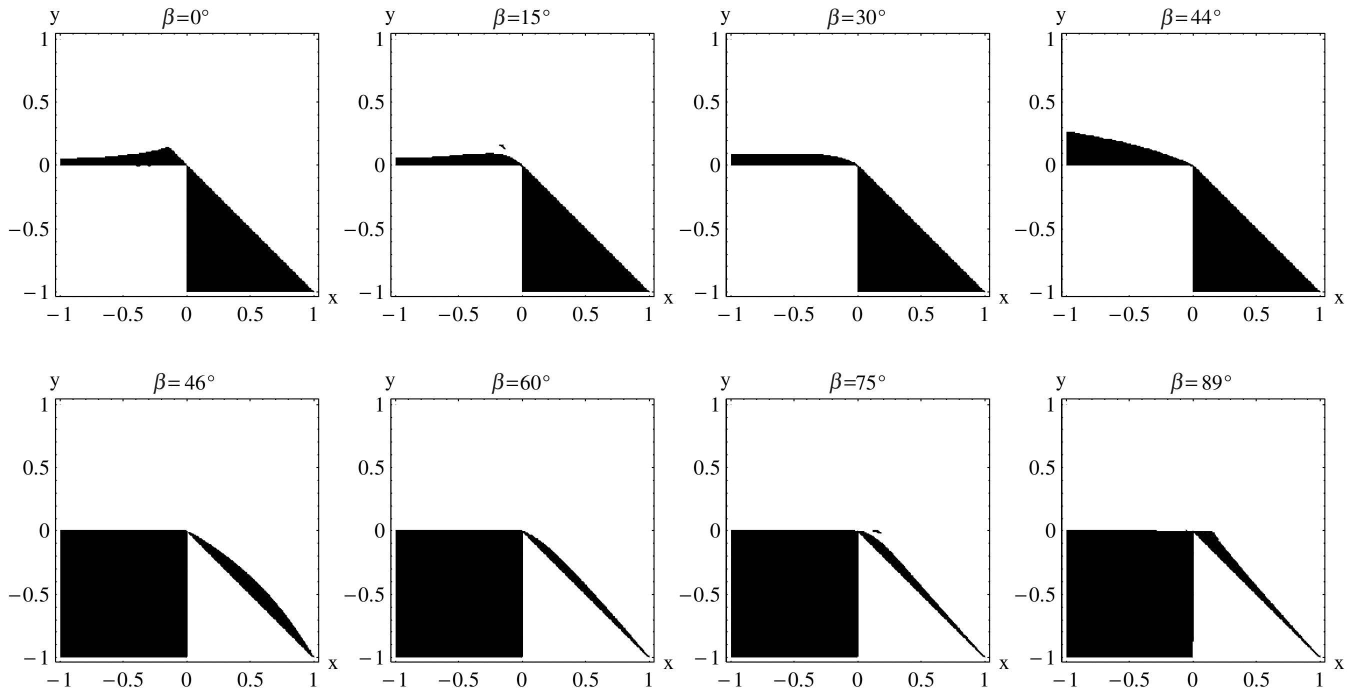


**Fig. 1 Regions of stability in the $(x, y)$ plane: case $\alpha_0 = \gamma_0 = 0$.**

$$
\begin{aligned}
a_0 &= \frac{(1-x)(1+y)}{(1+xy)^2} \\
a_2 &= -\frac{(1-x)(1+y)}{2(1+xy)^3}[-2+6x+3y-2xy+3x^2y-y^2 \\
&\quad + x^2y^2 + (9y+2xy-3x^2y+y^2+6xy^2+x^2y^2)\cos 2\gamma_0] \\
a_4 &= \frac{(1-x)(1+y)}{2(1+xy)^3}[-6x-3y-3x^2y+7y^2-7x^2y^2 \\
&\quad + (-3y+10xy+9x^2y+5y^2+6xy^2+5x^2y^2)\cos 2\gamma_0 \\
&\quad + 6y^2(1-x^2)\cos 4\gamma_0] \\
a_6 &= \frac{12(1-x)(1+y)(x+y)xy}{(1+xy)^3}\cos 2\gamma_0
\end{aligned} \tag{33}
$$

Figure 2 shows the results of the analysis of conditions (11–14) for the indicated values of $\gamma_0$ in the plane $(x, y)$ (the $x$ axis is horizontal, and the $y$ axis is vertical). Regions of stability exist for all $\gamma_0$ from the interval $0 \le \gamma_0 < \pi/2$ except for $\gamma_0 = \pi/4$ (for this value of roll angle $\gamma_0$ the coefficient $a_6$ vanishes and the respective equilibrium is unstable). Both for $0 < \gamma_0 < \pi/4$ and for $\pi/4 < \gamma_0 < \pi/2$ the evolution of the stability regions is fairly regular. A significant change occurs when $\gamma_0$ crosses the value $\gamma_0 = \pi/4$ which is related to the change of the sign of $\cos 2\gamma_0$ in expressions (33). When $\gamma_0$ belongs to one of the above intervals, stability regions exist in two geometrically different situations: 1) when the axis of the satellite's medium inertia moment is aligned with the tangent to the orbit and the axis of the maximum moment forms an angle less than $\pi/4$ with the normal to the orbit plane; 2) when the axis of the maximum moment of inertia is aligned with the tangent to the orbit, and the axis of the minimum moment makes an angle less then $\pi/4$ with the normal to the orbit plane. In both cases, additional conditions have to be satisfied.

## Special Case II

The stability can be studied analytically if one of the satellite's principal axes belongs to a coordinate plane of the orbital frame. In this case one of the orientation angles is equal to $\pi n/2$, $n \in Z$. The system of Eqs. (7) and (8) then has an explicit solution that can be substituted into Eqs. (11–14) to verify stability. It is not difficult to see that it suffices to study only the case $n = 0$, because other orientations of this type can be reduced to it by relabeling the reference axes. The two remaining orientation angles should be different from $\pi m/2$, $m \in Z$, otherwise the orientation in question is reduced to special case I.

Both for a satellite with general mass distribution and for thin plates it can be shown that the conditions of stability are incompatible, so these equilibrium orientations are unstable. The proof is rather cumbersome (it is included in Appendix B).

## General Case

To study stability of an equilibrium in the general case, we transform the characteristic equation using new variables

$$s = \tan \alpha_0, \qquad p = \tan \gamma_0$$

The coefficients of the resulting equation are given in Appendix A.

Note that Eqs. (7) and (8) are algebraic with respect to variables $s$ and $p$, so we can eliminate the variable $s$ from Eqs. (7) and (8) using the method of resultant. We get a polynomial $P(p)$ of degree 14. Each real root of this polynomial corresponds to a unique real solution $(s, p)$ of Eqs. (7) and (8).

We use this property in the following numerical procedure. First, we choose some value of the angle $\beta_0$. Next, we find all roots of the polynomial $P(p)$ for the values of the inertia parameters $x$ and $y$ within the square (19). For each root we find the corresponding solution $(s, p)$ of Eqs. (7) and (8), and thus the angles $\alpha_0$ and $\gamma_0$ which satisfy this system. Finally we verify stability conditions (11–14) for all these pairs $(\alpha_0, \gamma_0)$. This analysis was performed for 5 deg $\le \beta_0 \le 85$ deg with $\Delta\beta_0 = 5$ deg. It did not reveal any points in the space of parameters $(\alpha_0, \beta_0, \gamma_0, x, y)$ where the conditions of stability are satisfied. This does not prove the absence of stable equilibria in this case, but limits their localization to some small domains in the parameter space.

## Conclusions

We study equilibria of a satellite in a circular orbit under the action of the gravitational and constant torques. The gravitational torque arises when the satellite is not spherically symmetric. A constant torque can either be produced actively or result from gas or fuel leakages. The components of constant torque corresponding to a specific equilibria with respect to the orbital reference frame can be easily found as functions of orientation angles. In this case, the

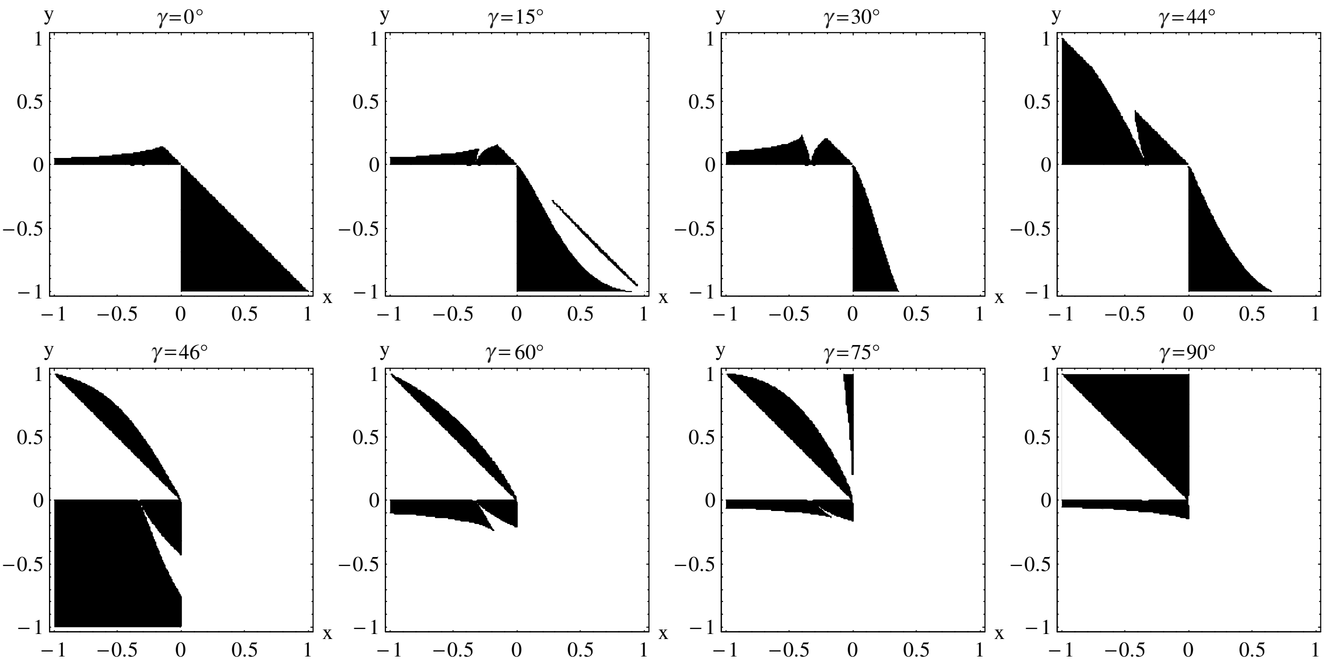


**Fig. 2 Regions of stability in the $(x, y)$ plane: case $\alpha_0 = \beta_0 = 0$.**

equations of the satellite's motion do not admit any integral of motion, so stability of an equilibrium is examined by studying the respective characteristic equation.

If a satellite in a circular orbit is subject only to the gravity-gradient torque, it is well known that the necessary conditions of equilibrium are satisfied in two cases: 1) the axis of its greatest inertia moment coincides with the normal to the orbit and the axis of the smallest inertia moment is aligned with the local vertical; 2) the axis of the satellite's greatest inertia moment is aligned with the tangent to the orbit and the axis of the smallest inertia moment coincides with the normal to the orbit plane. All stable orientations found in the present paper for a satellite with arbitrary mass distribution can be obtained from one of the above configurations by rotation either around the tangent to the orbit or around the local vertical by an angle smaller than $\pi/4$.

More specifically, the necessary conditions of stability are satisfied only if one of the orbital frame's axes coincides with a central principal axis of inertia. We show that they are always satisfied in the following two cases: 1) when the axis of the satellite's minimal moment of inertia is directed along the local vertical, whereas the angle between the axis of its maximal moment of inertia and the normal to the orbit plane does not exceed 45 deg (that is, the axis of its maximal moment of inertia is inclined toward the normal to the orbit); 2) when the satellite is a thin plate situated in the orbit plane and the axis of its minimal moment of inertia is inclined toward the local vertical. The stability regions also exist in three more situations: 1) the axis of medium moment is aligned to the local vertical and the axis of maximal inertia moment is inclined toward the tangent to the orbit; 2) the axis of the satellite's medium inertia moment is aligned with the tangent to the orbit and the axis of the maximal moment is inclined toward the normal to the orbit plane; 3) the axis of the maximal moment of inertia is aligned with the tangent to the orbit, and the axis of the minimal moment is inclined toward the normal to the orbit. In the latter three cases, additional conditions have to be satisfied.

If only one of the central principal axes of the satellite is situated in a coordinate plane of the orbital reference frame, an equilibrium is unstable because the necessary conditions of stability are proved to be incompatible.

In the general case, none of the principal axes of inertia lies in the coordinate planes of the orbital reference frame. For this case, numerical study was performed; it did not reveal any stable equilibrium configuration.

The above analysis of equilibrium configurations may prove useful for mission planning because even an unstable equilibrium orientation can be maintained by fairly small control efforts, and a stable equilibrium minimizes the control requirements.

## Appendix A

Using the notation

$$s = \tan\alpha_0, \qquad p = \tan\gamma_0, \qquad x = \frac{B-A}{C}, \qquad y = \frac{C-B}{A}$$

one can write characteristic equation (6) as

$$(\tilde{a}_0\lambda^6 + \tilde{a}_2\lambda^4 + \tilde{a}_3\lambda^3 + \tilde{a}_4\lambda^2 + \tilde{a}_5\lambda + \tilde{a}_6)\Phi(p, s, x, y)\cos\beta_0 = 0$$

where

$$\tilde{a}_0 = -4(1+p^2)^2(1+s^2)^2(1+xy)$$

$$\begin{aligned}\tilde{a}_2 = {} & -2(1+p^2)(1+s^2)\{2 + 2p^2 + 2s^2 + 2p^2s^2 - 6x \\ & - 6p^2x + 3s^2x + 3p^2s^2x + x^2 + p^2x^2 + s^2x^2 + p^2s^2x^2 \\ & - 12y + 6p^2y + 6s^2y - 3p^2s^2y + 2xy + 4p^2xy + 2s^2xy \\ & + 4p^2s^2xy - 6p^2x^2y + 3p^2s^2x^2y + p^2y^2 + p^2s^2y^2 - 6xy^2 \\ & + 6p^2xy^2 + 3s^2xy^2 - 3p^2s^2xy^2 - x^2y^2 - s^2x^2y^2 \\ & + [-x^2 - p^2x^2 - 2xy + p^2y^2 - x^2y^2 + s^2(9x + 9p^2x \\ & - x^2 - p^2x^2 + 9p^2y - 2xy + 6x^2y + 3p^2x^2y + p^2y^2 - 3xy^2 \\ & + 3p^2xy^2 - x^2y^2)]\cos 2\beta_0 + 12psy(3 - x^2 + 2xy)\sin\beta_0\}\end{aligned}$$

$$\begin{aligned}\tilde{a}_3 = {} & -2(1+p^2)(1+s^2)\{-3sx - 3p^2sx - 6sy + 3p^2sy \\ & - 3p^2sx^2y - 3sxy^2 + 3p^2sxy^2 + 3s[-p^2y + (-2+p^2)x^2y \\ & + x(-1 - p^2 - y^2 + p^2y^2)]\cos 2\beta_0 + py(-6 + 6s^2 \\ & + 7x^2 - 2s^2x^2 + xy + 4s^2xy)\sin\beta_0 + (px^2y + 4ps^2x^2y \\ & + pxy^2 + 4ps^2xy^2)\sin 3\beta_0\}\end{aligned}$$

$$\tilde{a}_4 = -(1+p^2)^2 x(-6+6s^4+8x-5s^2x+14s^4x) - 2(1+p^2)\{-3(-2+p^2)(-1+s^4)+[-18(-1+s^2) + p^2(-10+13s^2+14s^4)]x+3p^2(-1+s^4)x^2\}y+\{-36 + 40p^2-8p^4+36s^2-166p^2s^2+5p^4s^2+10p^2s^4 - 14p^4s^4+6(-1+p^4)(-1+s^4)x-[-8+5s^2-14s^4 + 36p^4(-1+s^2)+2p^2(20-83s^2+5s^4)]x^2\}y^2 - 2\{(1+p^2)^2x[-3-4x-17s^2x+s^4(3+5x)] + (1+p^2)[3p^2(-1+s^4)+2(-4-17s^2+5s^4)x + 3(-2+3p^2)(-1+s^4)x^2]y+[p^2(2+97s^2-13s^4)(x^2-1) + x(9-4x-17s^2x-9s^4+5s^4x)+p^4(4+17s^2-9x - 5s^4+9s^4x)]y^2\}\cos 2\beta_0 - 3s^2(1+4s^2)[x(1+p^2-y) + p^2y][p^2y+x(1+p^2+y)]\cos 4\beta_0+12psy\{-(1+p^2)(4 + 4s^2-11x-5s^2x+2(1+s^2)x^2)-[-18+7p^2+6x + x(6p^2-18p^2x+7x)+s^2(6+6x-11x^2-11p^2 + 6p^2x+6p^2x^2)]y\}\sin\beta_0+12psy\{(1+p^2)x(-1+2x - 7s^2+2s^2x)-[p^2(1+7s^2-2s^2x-2x)+x(-2+x - 2s^2+7s^2x)]y\}\sin 3\beta_0$$

$$\tilde{a}_5 = -3s\{2(1+p^2)[6+p^2+(-6+4p^2)s^2]xy+[12-46p^2 + 7p^4-2(6-13p^2+p^4)s^2]y^2+x^2[7+14p^2+7p^4 - 7y^2+46p^2y^2-12p^4y^2+2s^2(-1-2p^2-p^4+y^2 - 13p^2y^2+6p^4y^2)]\}+6s[6(1+p^2)(-1+s^2)xy + p^2(-13+3p^2+23s^2-3p^2s^2)y^2+x^2(-3-6p^2 - 3p^4+3s^2+6p^2s^2+3p^4s^2-3y^2+13p^2y^2+3s^2y^2 - 23p^2s^2y^2)]\cos 2\beta_0+3s(1+4s^2)(x^2+2p^2x^2+p^4x^2 + 2p^2xy+2p^4xy+p^4y^2-x^2y^2)\cos 4\beta_0-2py\{-3[-6 + 7p^2+(36-27p^2)s^2+2(-3+p^2)s^4]y+(1+p^2)x[-3 + 4y+4s^4(3+y)+s^2(-27+8y)]+x^2[4+4p^2-21y + 18p^2y+s^2(8+8p^2+81y-108p^2y)+s^4(4+4p^2 - 6y+18p^2y)]\}\sin\beta_0-2py\{3p^2(-1-9s^2+4s^4)y + x^2[4+4p^2+s^2(8+8p^2-27y)-3y+4s^4(1+p^2 + 3y)]+(1+p^2)x[-3+4y+4s^4(3+y) + s^2(-27+8y)]\}\sin 3\beta_0$$

$$\tilde{a}_6 = -24(1+p^2)(1+s^2)xy(x+y)[2(-1+p^2)(-1 + s^2)\cos 2\beta_0+ps(5\sin\beta_0-3\sin 3\beta_0)]$$

## Appendix B

Here we consider in detail special case II.

### Special Case IIa: $\sin 2\alpha_0 = 0$, $\sin 2\beta_0 \neq 0$, $\sin 2\gamma_0 \neq 0$

We first examine the case of an equilibrium orientation with the pitch angle $\alpha_0 = 0$. The torque that causes this orientation is

$$a = -(3+\cos^2\beta_0)\sin\gamma_0\cos\gamma_0, \qquad b = -\sin\beta_0\cos\beta_0\sin\gamma_0$$
$$c = \sin\beta_0\cos\beta_0\cos\gamma_0$$

We begin with the analysis of a satellite with a three-dimensional mass distribution. The case of a thin plate will be considered separately.

#### Satellite with General Mass Distribution

Assuming that the satellite is not a thin plate and does not possess any kind of dynamical symmetry, one can use the inertial parameters $x$ and $y$. Equations (7) and (8) become

$$-\frac{y(x-1)(y+1)}{4(1+xy)^3}[-3+4x^2+xy + x(x+y)\cos 2\beta_0]\sin 2\beta_0\sin 2\gamma_0 = 0 \tag{B.1}$$

$$\frac{y(x-1)(y+1)}{16(1+xy)^3}\{4[3y+x(6-8y-8x+3xy)]\cos^2\beta_0 + 6y(x^2-1)(7+\cos 2\beta_0)\cos 2\gamma_0\}\sin 2\beta_0\sin 2\gamma_0 = 0 \tag{B.2}$$

One can calculate $\cos 2\beta_0$ and $\cos 2\gamma_0$ from this system, getting

$$\cos 2\beta_0 = \frac{3-4x^2-xy}{x(x+y)}, \qquad \cos 2\gamma_0 = \frac{6x-8x^2+3y-8xy+3x^2y}{3(y+x^2y+2xy^2)} \tag{B.3}$$

The inertial parameters $x$ and $y$ should satisfy inequalities (19), conditions of stability (11–14) and the obvious conditions

$$|\cos 2\beta_0| \le 1, \qquad |\cos 2\gamma_0| \le 1 \tag{B.4}$$

Let us show that these conditions are incompatible.

Substituting (B.3) to (B.4), one gets the following system of inequalities:

$$\frac{-3+5x^2+2xy}{x(x+y)} > 0 \tag{B.5}$$

$$-\frac{3(x-1)(x+1)}{x(x+y)} > 0 \tag{B.6}$$

$$\frac{2x(-3+4x+4y+3y^2)}{3y(1+x^2+2xy)} > 0 \tag{B.7}$$

$$\frac{2(x+y)(3-4x+3xy)}{3y(1+x^2+2xy)} > 0 \tag{B.8}$$

Condition (12) is now in the form

$$\frac{(x-1)^3(y+1)^2 f(x,y)}{2(1+xy)^5(1+x^2+2xy)} > 0 \tag{B.9}$$

where

$$f(x,y) = 7+23x-11x^2-3x^3+8y-12xy+24x^2y - 4x^3y-3y^2+gxy^2-9x^2y^2+3x^3y^2 \tag{B.10}$$

According to Eq. (19),

$$1 \pm x > 0, \qquad 1 \pm y > 0, \qquad 1+xy > 0$$

and because $y^2 < 1$,

$$1+x^2+2xy > 1+x^2y^2+2xy = (1+xy)^2 > 0$$

So the system of Eqs. (B.5), (B.6), (B.7), (B.8), and (B.9) can be transformed to

$$-3+5x^2+2xy > 0 \tag{B.11}$$

$$x(x+y) > 0 \tag{B.12}$$

$$xy(-3+4x+4y+3y^2) > 0 \tag{B.13}$$

$$y(x+y)(3-4x+3xy) > 0 \tag{B.14}$$

$$f(x,y) < 0 \tag{B.15}$$

If $x < 0$ then (B.12) leads to $x + y < 0$, and from (B.13) we have

$$-3+4x+4y+3y^2 = 4(x+y) - 3(1-y^2) < 0$$

so $y$ should be positive. Relations (B.11) and (B.14) then yield

$$-3+5x^2+2xy > 0, \qquad 3-4x+3xy < 0$$

Excluding the product $xy$ leads to

$$15x^2+8x-15 > 0$$

This inequality cannot be satisfied for $-1 < x < 0$.

If $x > 0$, then $x + y > 0$ by Eq. (B.12). A straightforward calculation locates the minimum of function (B.10) on the domain

$$0 \le x \le 1, \qquad -1 \le y \le 1, \qquad x+y \ge 0 \tag{B.16}$$

This minimum equals zero and is achieved at the point $(1, -1)$ of the boundary. Thus $f(x, y)$ is positive in the interior of domain (B.16), which contradicts Eq. (B.15). Hence the conditions of stability are incompatible, and all equilibria with $\alpha_0 = 0$, $\beta_0 \neq 0$, and $\gamma_0 \neq 0$ are unstable.

**Thin Plate**

The analysis of the stability of a satellite in the shape of a thin plate is performed using inertial parameters $\theta_A$, $\theta_C$. Taking into account that $\sin 2\beta_0 \neq 0$ and $\sin 2\gamma_0 \neq 0$, Eqs. (7) and (8) can be written as

$$\left[4\theta_A^2+\theta_C-4\theta_A(1+\theta_C)\right] + (\theta_A-1)(\theta_A-\theta_C)\cos 2\beta_0 = 0$$

$$\left[-3+8\theta_A^2+(8-3\theta_C)\theta_C-5\theta_A(1+\theta_C)\right](1+\cos 2\beta_0)$$
$$-3(\theta_A-\theta_C-1)(\theta_C-1)(7+\cos 2\beta_0)\cos 2\gamma_0 = 0$$

One can find $\cos 2\beta_0$ and $\cos 2\gamma_0$ from this system:

$$\cos 2\beta_0 = \frac{4\theta_A^2+\theta_C-4\theta_A(1+\theta_C)}{(1-\theta_A)(\theta_A-\theta_C)} \tag{B.17}$$

$$\cos 2\gamma_0 = \frac{-3+8\theta_A^2+(8-3\theta_C)\theta_C-5\theta_A(1+\theta_C)}{3(1-\theta_C)[\theta_A^2+2\theta_C-\theta_A(1+\theta_C)]}\theta_A \tag{B.18}$$

Further study depends on the position of the plate in the IO.

**Case $A = B + C$**

In this case, the plate in the IO is orthogonal to the tangent to the orbit. Substituting $1 + \theta_C - \theta_A = 0$ to Eq. (B.17), one obtains $\cos 2\beta_0 = -1$ that corresponds to special case Ic.

**Case $B = A + C$**

In this case the plate in the IO is situated in the orbit plane. Simplifying relations (B.17) and (B.18) gives

$$\cos 2\beta_0 = \frac{7-8\theta_C}{-1+2\theta_C}, \qquad \cos 2\gamma_0 = \frac{5}{3} - \frac{4}{3\theta_C}$$

Substituting these expressions to the formulas for $a_0$ and $a_2$ leads to

$$a_0 a_2 = 4(\theta_C-2)(1-\theta_C)^3\theta_C\cos^2\beta_0$$

As soon as $\cos\beta_0 > 0$ and $\theta_C < 1$, $a_0 a_2$ is always negative, and the respective equilibrium is unstable.

**Case $C = A + B$**

Now we consider the plate that in the IO is orthogonal to the local vertical. In this case $\theta_A - \theta_C + 1 = 0$ and relation (B.17) is reduced to

$$\cos 2\beta_0 = \frac{8-7\theta_C}{-2+\theta_C}$$

Then

$$1+\cos 2\beta_0 = \frac{6(1-\theta_C)}{\theta_C-2} < 0$$

so $\cos 2\beta_0 < -1$ and this solution does not exist.

Thus we have shown that all equilibria that correspond to special case IIa are unstable.

**Special Case IIb: $\sin 2\alpha_0 \neq 0$, $\sin 2\beta_0 = 0$, $\sin 2\gamma_0 \neq 0$**

Consider the case $\beta_0 = 0$. Then for a nonsymmetrical satellite different from a thin plate Eqs. (7) and (8) can be written in the form

$$\frac{3(x-1)(y+1)}{4(1+xy)^3}[2x+y+x^2y$$
$$+ y(1+x^2+2xy)\cos 2\gamma_0]\sin 2\alpha_0 = 0,$$
$$\frac{3(x-1)(y+1)}{32(1+xy)^3}\{6\cos 2\alpha_0[-8x^2-8xy+3(x^2-1)y^2$$
$$-4y(y+2x+x^2y)\cos 2\gamma_0 + (x^2-1)y^2\cos 4\gamma_0]$$
$$+20(1-x^2)y^2\sin^2 2\gamma_0\}\sin 2\alpha_0 = 0$$

It permits one to calculate $\cos 2\gamma_0$ and $\cos 2\alpha_0$:

$$\cos 2\gamma_0 = -\frac{2x+y+x^2y}{y(1+x^2+2xy)}, \qquad \cos 2\alpha_0 = \frac{5(1+xy)}{3x(x+y)}$$

The conditions of existence for these trigonometric functions lead to the following system of inequalities:

$$\frac{-5+3x^2-2xy}{3x(x+y)} > 0 \tag{B.19}$$

$$\frac{5+3x^2+8xy}{3x(x+y)} > 0 \tag{B.20}$$

$$\frac{2(x+y)(1+xy)}{y(1+x^2+2xy)} > 0 \tag{B.21}$$

$$\frac{2x(-1+y)(1+y)}{y(1+x^2+2xy)} > 0 \tag{B.22}$$

This system is incompatible for $x$ and $y$ that satisfy conditions (19). Indeed, it was shown before that $1 + x^2 + 2xy > 0$ so from Eq. (B.22) it follows that $xy < 0$, and Eq. (B.21) implies that $y(x + y) > 0$, consequently $x(x + y) < 0$. Then from Eq. (B.20) one gets $5 + 3x^2 + 8xy < 0$. But $y^2 < 1$, so $5 + 3x^2 + 8xy > 5 + 3x^2y^2 + 8xy > 0$ for $xy > -1$. Thus, in this case all equilibrium orientations are unstable.

It can be easily proved that all equilibrium orientations of a thin plate are unstable too.

**Special Case IIc: $\sin 2\alpha_0 \neq 0, \sin 2\beta_0 \neq 0, \sin 2\gamma_0 = 0$**

Conditions (7) and (8) for $\gamma_0 = 0$ can be expressed as

$$\frac{3(x-1)(y+1)}{4(1+xy)^3}[x + 2y + xy^2 + x(1 + y^2 + 2xy)\cos 2\beta_0] \sin 2\alpha_0 \cos\beta_0 = 0$$

$$\frac{3(x-1)(y+1)}{32(1+xy)^3}\{6\cos 2\alpha_0[-8y^2 - 8xy + 3(y^2-1)x^2 - 4x(2y + x + xy^2)\cos 2\beta_0 + (y^2-1)x^2\cos 4\beta_0] + 20(y^2-1)x^2\sin^2 2\beta_0\} \sin 2\alpha_0 \cos\beta_0 = 0$$

so one can calculate $\cos 2\beta_0$ and $\cos 2\alpha_0$:

$$\cos 2\beta_0 = -\frac{x + 2y + xy^2}{x(1 + 2xy + y^2)}, \qquad \cos 2\alpha_0 = -\frac{5(1+xy)}{3y(x+y)} \tag{B.23}$$

Expressions (B.23) coincide with those obtained in special case IIb for $\cos 2\gamma_0$ and $\cos 2\alpha_0$, respectively (it suffices to substitute $x \to y$, $y \to -x$). Hence the conditions of existence for $\cos 2\gamma_0$ and $\cos 2\alpha_0$ in this case are also incompatible, and equilibrium orientations are unstable. The same result is obtained in the case of a thin plate.

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